NWI-MOL155
Quantum Mechanics

Exercises week 5

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November 1, 2024

Exercise 1 Eigenfunctions and stationary states

For a time-independent potential, $V(x,t) = V(x)$, the Schrödinger equation can be separated in time- and position-dependent parts. The position dependent part, the *time-independent Schrödinger equation* is an eigenvalue problem:

$$\hat{\mathsf{H}}\,|n\rangle = E_n\,|n\rangle$$

We have learned in the Lecture that every wavefunction $|\psi\rangle$ can be expressed in terms of the eigenfunctions $\{|n\rangle\}$ as $|\psi\rangle = \sum_n c_n\,|n\rangle$.

1. Calculate the probability $P_n = |\,\langle n|\psi\rangle\,|^2$ to measure an energy E_n, and find values for c_n.
2. Show that $\langle\hat{\mathsf{A}}\rangle = \langle\psi|\hat{\mathsf{A}}|\psi\rangle$ can be written in terms of the probability P_n.
3. The time dependent wavefunctions $\Psi(x,t)$ that are made up of only <u>one</u> solution of the time-independent Schrödinger equation, $\Psi(x,t) = |n\rangle\, e^{-iE_n t/\hbar}$ are so-called *stationary states.*
   a) Show that for stationary states, as opposed to mixed states, the probability function is independent of time
   $$|\Psi(x,t)|^2 = |\psi(x)|^2.$$
   b) Show that for stationary states, the expectation value of any observable Q is independent of time:
   c) Calculate the variance for the energy $\Delta E = \sqrt{\langle\hat{\mathsf{H}}^2\rangle - \langle\hat{\mathsf{H}}\rangle^2}$ for a stationary state. Comment on the result.

Exercise 2 Scattering off a potential step

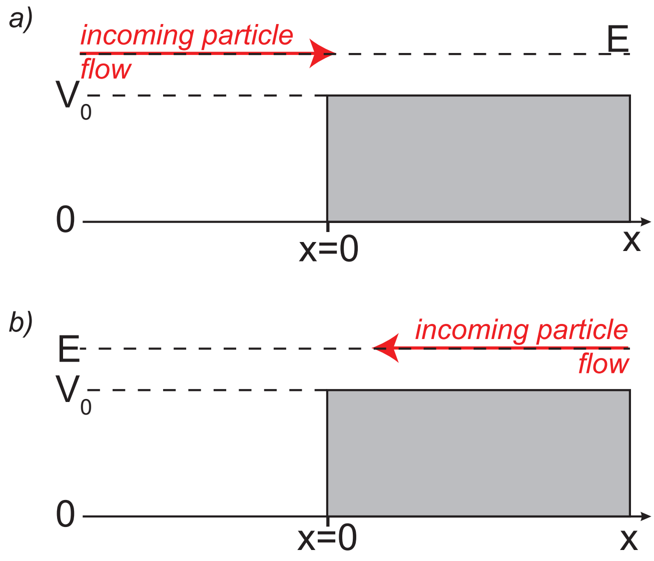
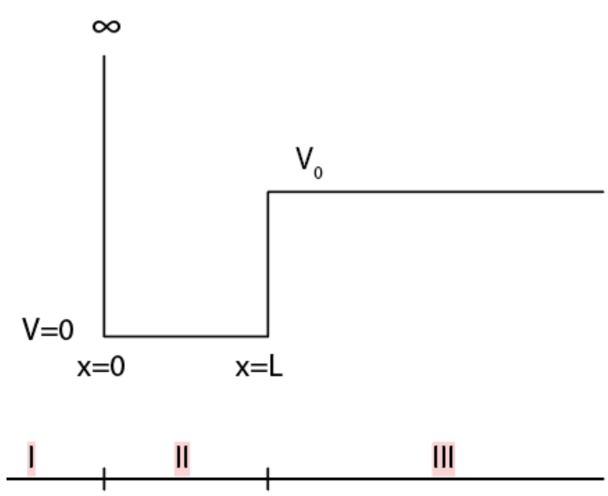


Figure 1:

In the lecture we treated the problem of a particle scattering off (and through) a potential step with $E < V_0$. Figure 1 sketches a problem of a particle scattering off a potential step, with $E > V_0$.

1. Suppose the particle is incoming from the left side, Figure 1a. Write down the general solutions for the wavefunctions $\psi_I(x)$ and $\psi_{II}(x)$ for regions I and II, respectively. Argue whether we can already eliminate some of the terms in the general solution based upon the experiment under study, and write down the remaining boundary conditions.
2. Calculate the reflection and transmission coefficients: $R = \frac{|B|^2}{|A|^2}$ and $T = 1 - R$.
3. One would be tempted to use the more 'logical' expression $T = \frac{|C|^2}{|A|^2}$ for the transmission coefficient. Explain why this is not the case.
4. Suppose now the particle is coming from the right hand side, Figure 1b. Solve the scattering problem for this new situation and give the transmission and reflection coefficients. Note that reflection will be non-zero!. Discuss the differences between scattering from the right and left hand sides.

Exercise 3 α-decay: bound states and a finite length potential barrier



We consider a potential as shown in the figure above, including a potential barrier of width w. Throughout the whole exercise we consider a situation where $E < V_0$. We initially assume that the width of the barrier w is infinite.

1. Give the general solutions of the Schrödinger equation for regions I, II, and III. .
2. The general solution contains one term that can directly be eliminated based on the postulates of quantum mechanics. Which? Motivate your answer!
3. Write down the boundary conditions for this problem.
4. Show that the expression

   $$\tan(kL) = -\sqrt{\frac{E}{V_0 - E}}$$

   with $k = \sqrt{\frac{2mE}{\hbar^2}}$ dictates the solutions for the Schrödinger equation in the range $x = 0$ to $x = L$ for bound states ($E < V_0$)
5. For low energies ($E \ll V_0$) the right hand term becomes very small. Make a sketch of wavefunctions of the three lowest energy bound states of this system and motivate your choice.

We now reduce the width of the barrier to a finite length w. The current potential is an approximation for α particle decay, where a heavy nucleus decays into a lighter nucleus under the emission of an α-particle.

6. Give the general solutions for the Schrödinger equation in regions II, III and IV, and give four equations for the boundary conditions at positions $x = L$ and $x = L + w$.
7. Your solutions in the previous have in total six free parameters (the normalization factors). Use your knowledge about the wavefunctions in region II and the physical interpretation of α-decay to eliminate two of these parameters. Motivate each choice.
8. Sketch the wavefunctions for the new situation.